

An $O(\log k \log^2 n)$ -competitive Randomized Algorithm for the k -Sever Problem

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Abstract

In this paper, we show that there is an $O(\log k \log^2 n)$ -competitive randomized algorithm for the k -sever problem on any metric space with n points, which improved the previous best competitive ratio $O(\log^2 k \log^3 n \log \log n)$ by Nikhil Bansal et al. (FOCS 2011, pages 267-276).

Keywords: k -sever problem; Online algorithm; Primal-Dual method; Randomized algorithm;

1 Introduction

The k -sever problem is to schedule k mobile servers to serve a sequence of requests in a metric space with the minimum possible movement distance. In 1990, Manasse et al. introduced the k -sever problem as a generalization of several important online problems such as paging and caching problems [29] (Its conference version is [28]), in which they proposed a 2-competitive algorithm for the 2-sever problem and a $n - 1$ -competitive algorithm for the $n - 1$ sever problem in a n -point metric space. They still showed that any deterministic online algorithm for the k -sever problem is of competitive ratio at least k . They proposed the well-known k -sever conjecture: for the k -sever problem on any metric space with more than k different points, there exists a deterministic online algorithm with competitive ratio k .

It was in [29] shown that the k -sever conjecture holds for two special cases: $k = 2$ and $n = k + 1$. The k -sever conjecture also holds for the k -sever problem on a uniform metric. The special case of the k -sever problem on a uniform metric is called the paging (also known as caching) problem. Slator and Tarjan have proposed a k -competitive algorithm for the paging problem [31]. For some other special metrics such as line, tree, there existed k -competitive online algorithms. Yair Bartal

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and Elias Koutsoupias show that the Work Function Algorithm for the k -sever problem is of k -competitive ratio in the following special metric spaces: the line, the star, and any metric space with $k + 2$ points [16]. Marek Chrobak and Lawrence L. Larmore proposed the k -competitive Double-Coverage algorithm for the k -sever problem on trees [21].

For the k -sever problem on the general metric space, the k -sever conjecture remain open. Fiat et al. were the first to show that there exists an online algorithm of competitive ratio that depends only on k for any metric space: its competitive ratio is $\Theta((k!)^3)$. The bound was improved later by Grove who showed that the harmonic algorithm is of competitive ratio $O(k2^k)$ [25]. The result was improved to $(2^k \log k)$ by Y. Bartal and E. Grove [14]. A significant progress was achieved by Koutsoupias and Papadimitriou, who proved that the work function algorithm is of competitive ratio $2k - 1$ [27].

Generally, people believe that randomized online algorithms can produce better competitive ratio than their deterministic counterparts. For example, there are several $O(\log k)$ -competitive algorithms for the paging problem and a $\Omega(\log k)$ lower bound on the competitive ratio in [24, 30, 1, 4]. Although there were much work [17, 13, 15], the $\Omega(\log k)$ lower bound is still best lower bound in the randomized case. Recently, N. Bansal et al. propose the first polylogarithmic-competitive randomized algorithm for the k -sever problem on a general metrics space [3]. Their randomized algorithm is of competitive ratio $O(\log^2 k \log^3 n \log \log n)$ for any metric space with n points, which improves on the deterministic $2k - 1$ competitive ratio of Koutsoupias and Papadimitriou whenever n is sub-exponential k .

For the k -server problem on the general metric space, it is widely conjectured that there is an $O(\log k)$ -competitive randomized algorithm, which is called as the randomized k -server conjecture. For the paging problem (it corresponds to the k -sever problem on a uniform metric), there is $O(\log k)$ -competitive algorithms [24, 30, 1]. For the weighted paging problem (it corresponds to the k -sever problem on a weighted star metric space), there were also $O(\log k)$ -competitive algorithms [4, 9] via the online primal-dual method. More extensive literature on the k -server problem can be found in [26, 18].

In this paper, we show that there exists a randomized k -sever algorithm of $O(\log k \log^2 n)$ -competitive ratio for any metric space with n points, which improved the previous best competitive ratio $O(\log^2 k \log^3 n \log \log n)$ by Nikhil Bansal et al. [3].

In order to get our results, we use the online primal-dual method, which is developed by Buchbinder and Naor et al. in recent years. Buchbinder and Naor et al. have used the primal-dual method to design online algorithms for many online problems such as covering and packing problems, the ad-auctions problem and so on [4, 5, 6, 7, 8, 10]. First, we propose a primal-dual formulation for the fraction k -sever problem on a weighted hierarchical well-separated tree (HST). Then, we design an $O(\ell \log k)$ -competitive online algorithm for the fraction k -sever problem on a weighted HST with depth ℓ . Since any HST with n leaves can be transformed into a weighted HST with depth $O(\log n)$ with any leaf to leaf distance distorted by at most a constant [3], thus, we get an $O(\log k \log n)$ -competitive online algorithm for the fraction k -sever problem on an HST. Based on the known relationship between the fraction k -sever problem and the randomized k -sever problem, we get that there is an $O(\log k \log n)$ -competitive randomized algorithm for the k -sever problem on an HST with n points. By the metric embedding theory [22], we get that there is an $O(\log k \log^2 n)$ -competitive randomized algorithm for the k -sever problem on any metric space with n points.

2 Preliminaries

In this section, we give some basic definitions.

Definition 2.1. (Competitive ratio adapted from [32]) For a deterministic online algorithm $DALG$, we call it r -competitive if there exists a constant c such that for any request sequence ρ , $cost_{DALG}(\rho) \leq r \cdot cost_{OPT}(\rho) + c$, where $cost_{DALG}(\rho)$ and $cost_{OPT}(\rho)$ are the costs of the online algorithm $DALG$ and the best offline algorithm OPT respectively.

For a randomized online algorithm, we have a similar definition of competitive ratio:

Definition 2.2. (Adapted from [32]) For a randomized online algorithm $RALG$, we call it r -competitive if there exists a constant c such that for any request sequence ρ , $\mathbb{E}[cost_{RALG}(\rho)] \leq r \cdot cost_{OPT}(\rho) + c$, where $\mathbb{E}[cost_{RALG}(\rho)]$ is the expected cost of the randomized online algorithm $RALG$.

In order to analyze randomized algorithms for the k -sever problem, D. Türkouğlu introduce the fractional k -sever problem [32]. On the fractional k -sever problem, severers are viewed as fractional entities as opposed to units and an online algorithm can move fractions of servers to the requested point.

Definition 2.3. (Fractional k -sever problem adapted from [32]) Suppose that there are a metric space \mathcal{S} and a total of k fractional severers located at the points of the metric space. Given a sequence of requests, each request must be served by providing one unit server at requested point, through moving fractional servers to the requested point. The cost of an algorithm for servicing a sequence of requests is the cumulative sum of the distance incurred by each sever, where moving a w fraction of a server for a distance of δ costs $w\delta$.

In [11, 12], Bartal introduce the definition of a *Hierarchical Well-Separated Tree* (HST), into which a general metric can be embedded with a probability distribution. For any internal node, the distance from it to its parent node is σ times of the distance from it to its child node. The number σ is called the stretch of the HST. An HST with stretch σ is called a σ -HST. In the following, we give its formal definition.

Definition 2.4. (Hierarchically Well-Separated Trees (HSTs)[20]). For $\sigma > 1$, a σ -Hierarchically Well-Separated Tree (σ -HST) is a rooted tree $T = (V, E)$ whose edges length function d satisfies the following properties:

- (1). For any node v and any two children w_1, w_2 of v , $d(v, w_1) = d(v, w_2)$.
- (2). For any node v , $d(p(v), v) = \sigma \cdot d(v, w)$, where $p(v)$ is the parent of v and w is a child of v .
- (3). For any two leaves v_1 and v_2 , $d(p(v_1), v_1) = d(p(v_2), v_2)$.

Fakcharoenphol et al. showed the following result [22].

Lemma 2.5. If there is a γ -competitive randomized algorithm for the k -sever problem on an σ -HST with all requests at the n leaves, then there exists an $O(\gamma\sigma \log n)$ - competitive randomized online algorithm for the k -server problem on any metric space with n points.

We still need the definition of a weighted hierarchically well-separated tree introduced in [3].

Definition 2.6. (*Weighted Hierarchically Well-Separated Trees (Weighted HSTs)*[3]). A weighted σ -HST is a rooted tree satisfying the property (1), (3) of the definition 2.4 and the property: $d(p(v), v) \geq \sigma \cdot d(v, w)$ for any node v which is not any leaf or the root, where $p(v)$ is the parent of v and w is a child of v .

In [3], Banas et al. show that an arbitrary depth σ -HST with n leaves can be embedded into an $O(\log n)$ depth weighted σ -HST with constant distortion, which is described as follows.

Lemma 2.7. Let T be a σ -HST T with n leaves which is of possibly arbitrary depth. Then, T can be transformed into a weighted σ -HST \tilde{T} with depth $O(\log n)$ such that: the leaves of \tilde{T} and T are the same, and leaf to leaf distance in T is distorted in \tilde{T} by a at most $\frac{2\sigma}{\sigma-1}$ factor.

3 An $O(\log^2 k)$ Randomized Algorithm for the k -Sever Problem on an HST when $n = k + 1$

In this paper, we view the k -sever problem as the weighed caching problem such that the cost of evicting a page out of the cache using another page satisfies the triangle inequality, i.e., a point is viewed as a page; the set of k points that are served by k severers is viewed as the cache which holds k pages; the distance of two points i and j is viewed the cost of evicting the corresponding page p_i out of the cache using the corresponding page p_j .

Let $[n] = \{p_1, \dots, p_n\}$ denotes the set of n pages and $d(p_i, p_j)$ denotes the cost of evicting the page p_i out of the cache using the page p_j for any $p_i, p_j \in [n]$, which satisfies the triangle inequality: for any pages i, j, s , $d(p_i, p_i) = 0$; $d(p_i, p_j) = d(p_j, p_i)$; $d(p_i, p_j) \leq d(p_i, p_s) + d(p_s, p_j)$. Let p_1, p_2, \dots, p_M be the requested pages sequence until time M , where p_t is the requested page at time t . At each time step, if the requested page p_t is already in the cache then no cost is produced. Otherwise, the page p_t must be fetched into the cache by evicting some other pages p in the cache and a cost $\sum d(p, p_t)$ is produced.

In this section, in order to clearly describe our algorithm design idea, we consider the case $n = k + 1$.

First, we give some notations. Let σ -HST denote a hierarchically well-separated trees with stretch factor σ . Let N be the number of nodes in a σ -HST and leaves be p_1, p_2, \dots, p_n . Let $\ell(v)$ denote the depth of a node v . Let r denote the root node. Thus, $\ell(r) = 0$. For any leaf p , let ℓ denote its depth, i.e. $\ell(p) = \ell$. Let $p(v)$ denote the parent node of a node v . $C(v)$ denote the set of children of a node v . Let D denote the distance from the root to its a child. $D(v)$ denote the distance from a node v to its parent, i.e. $D(v) = d(v, p(v))$. It is easy to know that $D(v) = \frac{D}{\sigma^{\ell(v)-1}}$. Let T_v denote the subtree rooted at v and $L(T_v)$ denote the set of leaves in T_v . Let $|T_v|$ denote the number of the leaves in T_v . For a leaf p_i , let $A(p_i, j)$ denote the ancestor node of p_i at the depth j . Thus, $A(p_i, \ell)$ is p_i , $A(p_i, 0)$ is the root r and so on. At time t , let variable $x_{p_i, t}$ denote the fraction of p_i that is in the cache and $u_{p_i, t}$ denote the fraction of p_i that is out of cache. Obviously, $x_{p_i, t} + u_{p_i, t} = 1$ and $\sum_{p \in [n]} x_{p, t} = k$. For a node v , let $u_{v, t} = \sum_{p \in L(T_v)} u_{p, t}$, i.e., it is the total fraction of pages in the subtree T_v which is out of the cache. It is easy to see that $u_{v, t} = \sum_{w \in C(v)} u_{w, t}$. Suppose

that at time 0, the set of initial k pages in the cache is $I = \{p_{i_1}, \dots, p_{i_k}\}$.

At time t , when the request p_t arrives, if page p_t is fetched mass $\Delta(p_t, p)$ into the cache by evicting out the page p in the cache, then the evicting cost is $d(p, p_t) \cdot \Delta(p_t, p)$. For a σ -HST metric,

suppose the path from p_t to p in it is: $p_t, v_j, \dots, v_1, v, v'_1, \dots, v'_j, p$, where v is the first common ancestor node of p_t and p . By the definition of a σ -HST, we have $D(p_t) = D(p)$ and $D(v_i) = D(v'_i)$ for any $1 \leq i \leq j$. Thus, $d(p, p_t) = D(p_t) + \sum_{i=1}^j D(v_i) + D(p) + \sum_{i=1}^j D(v'_i) = 2D(p_t) + \sum_{i=1}^j 2D(v_i)$. So, the evicting cost is $(2D(p_t) + \sum_{i=1}^j 2D(v_i)) \cdot \Delta(p_t, p)$. Since p can be any page in $[n] \setminus \{p_t\}$, the evicting cost incurred at time t is $\sum_{v=1}^N 2D(v) \max\{0, u_{v,t-1} - u_{v,t}\}$. Thus, we give the LP formulation for the fractional k -sever problem on a σ -HST as follows.

<div style="display: flex; justify-content: space-between;"> <div style="flex: 1;"> <p>(P) Minimize $\sum_{t=1}^M \sum_{v=1}^N 2D(v) z_{v,t} + \sum_{t=1}^M \infty \cdot u_{p_t,t}$</p> <p>Subject to $\forall t > 0$ and $S \subseteq [n]$ with $S > k$, $\sum_{p \in S} u_{p,t} \geq S - k$;</p> <p style="margin-left: 40px;">$\forall t > 0$ and a subtree $T_v (v \neq r)$, $z_{v,t} \geq \sum_{p \in L(T_v)} (u_{p,t-1} - u_{p,t})$;</p> <p style="margin-left: 40px;">$\forall t > 0$ and node v, $z_{v,t}, u_{v,t} \geq 0$;</p> <p style="margin-left: 40px;">For $t = 0$ and any leaf node $p \in I$, $u_{p,0} = 0$;</p> <p style="margin-left: 40px;">For $t = 0$ and any leaf node $p \notin I$, $u_{p,0} = 1$;</p> </div> <div style="flex: 0 0 100px; text-align: right;"> <p>(3.1)</p> <p>(3.2)</p> <p>(3.3)</p> <p>(3.4)</p> <p>(3.5)</p> </div> </div>

The first primal constraint (3.1) states that at any time t , if we take any set S of vertices with $|S| > k$, then $\sum_{p \in S} u_{p,t} = |S| - \sum_{p \in S} x_{p,t} \geq |S| - \sum_{p \in [n]} x_{p,t} = |S| - k$, i.e., the total number of pages out of the cache is at least $|S| - k$. The variables $z_{v,t}$ denote the total fraction mass of pages in T_v that are moved out of the subtree T_v (Obviously, it is not needed to define a variable $z_{r,t}$ for the root node). The fourth and fifth constraints ((3.4) and (3.5)) enforce the initial k pages in the cache are p_{i_1}, \dots, p_{i_k} . The first term in the object function is the sum of the moved cost out of the cache and the second term enforces the requirement that the page p_t must be in the cache at time t (i.e., $u_{p_t,t} = 0$).

Its dual formulation is as follows.

<div style="display: flex; justify-content: space-between;"> <div style="flex: 1;"> <p>(D) Maximize $\sum_{t=1}^M \sum_{S \subseteq [n], S > k} (S - k) a_{S,t} + \sum_{p \notin I} \gamma_p$</p> <p>Subject to $\forall t$ and $p \in [n] \setminus \{p_t\}$, $\sum_{S: p \in S} a_{S,t} - \sum_{j=1}^{\ell} (b_{(A(p,j),t+1)} - b_{(A(p,j),t)}) \leq 0$</p> <p style="margin-left: 40px;">$\forall t = 0$ and $\forall p \in [n]$, $\gamma_p - \sum_{j=1}^{\ell} b_{(A(p,j),1)} \leq 0$</p> <p style="margin-left: 40px;">$\forall t > 0$ and any subtree T_v, $b_{v,t} \leq 2D(v)$</p> <p style="margin-left: 40px;">$\forall t > 0$ and v and $S > k$, $a_{S,t}, b_{v,t} \geq 0$</p> </div> <div style="flex: 0 0 100px; text-align: right;"> <p>(3.6)</p> <p>(3.7)</p> <p>(3.8)</p> <p>(3.9)</p> </div> </div>

In the dual formulation, the variable $a_{S,t}$ corresponds to the constraint of the type (3.1); the variable $b_{v,t}$ corresponds to the constraint of the type (3.2); The variable γ_p corresponds to the constraint of the type (3.4) and (3.5).

Based on above primal-dual formulation, we extend the design idea of Bansal et al.'s primal-dual algorithm for the metric task system problem on a σ -HST [10] to the k -sever problem on a σ -HST. The design idea of our online algorithm is described as follows. During the execution of our algorithm, it always maintains the following relation between the primal variable $u_{v,t}$ and dual variable $b_{v,t+1}$: $u_{v,t} = f(b_{v,t+1}) = \frac{|T_v|}{k} (\exp(\frac{b_{v,t+1}}{2D(v)} \ln(1+k)) - 1)$. When the request p_t arrives at time t , the page p_t is gradually fetched into the cache and other pages are gradually moved out of

the cache by some rates until p_t is completely fetched into the cache(i.e, $u_{p_t,t}$ is decreased at some rate and other $u_{p,t}$ is increased at some rate for any $p \in [n] \setminus \{p_t\}$ until $u_{p_t,t}$ becomes 0). It can be viewed that we move mass $u_{p_t,t}$ out of leaf p_t through its ancestor nodes and distribute it to other leaves $p \in [n] \setminus \{p_t\}$. In order to compute the exact distributed amount at each page $p \in [n] \setminus \{p_t\}$, the online algorithm should maintain the following invariants:

1. (**Satisfying Dual Constraints:**) It is tight for all dual constraints of type (3.6) on other leaves $[n] \setminus \{p_t\}$.
2. (**Node Identity Property:**) $u_{v,t} = \sum_{w \in C(v)} u_{w,t}$ holds for each node v .

We give more clearer description of the online algorithm process. At time t , when the request p_t arrives, we initially set $u_{p_t,t} = u_{p_t,t-1}$. If $u_{p_t,t} = 0$, then we do nothing. Thus, the primal cost and the dual profit are both zero. All the invariants continue to hold. If $u_{p_t,t} \neq 0$, then we start to increase variable a_S at rate 1. At each step, we would like to keep the dual constraints (3.6) tight and maintain the node identity property. However, increasing variable a_S violates the dual constraints(3.6) on leaves in $[n] \setminus \{p_t\}$. Hence, we increase other dual variables in order to keep these dual constraints (3.6) tight. But, increasing these variables may also violate the node identity property. So, it makes us to update other dual variables. This process results in moving initial $u_{p_t,t}$ mass from leaf p_t to leaves $[n] \setminus \{p_t\}$. We stop the updating process when $u_{p_t,t}$ become 0.

In the following, we will compute the exact rate at which we should move mass $u_{p_t,t}$ from p_t through its ancestor nodes at time t to other leaves in $[n] \setminus \{p_t\}$ in the σ -HST. Because of the space limit, we put proofs of the following some claims in the Appendix. First, we show one property of the function f .

Lemma 3.1. $\frac{du_{v,t}}{db_{v,t+1}} = \frac{\ln(1+k)}{2D(v)}(u_{v,t} + \frac{|T_v|}{k})$.

Proof. Since $u_{v,t} = \frac{|T_v|}{k}(\exp(\frac{b_{v,t+1}}{2D(v)} \ln(1+k)) - 1)$, we take the derivative over $b_{v,t+1}$ and get the claim. \square

In order to maintain the **Node Identity Property:** $u_{v,t} = \sum_{w \in C(v)} u_{w,t}$ for each node v at any time t , when $u_{v,t}$ is increased or decreased, it is also required to increase or decrease the children of v at some rate. The connection between these rates is given.

Lemma 3.2. For a node v , if we increase variable $b_{v,t+1}$ at rate h , then we have the following equality:

$$\frac{1}{\sigma}(u_{v,t} + \frac{|T_v|}{k}) \cdot \frac{db_{v,t+1}}{dh} = \sum_{w \in C(v)} \frac{db_{w,t+1}}{dh} \cdot (u_{w,t} + \frac{|T_v|}{k})$$

We need one special case of lemma 3.2: when the variable $b_{v,t+1}$ is increased (decreased) at rate h , it is required that the increasing (decreasing) rate of all children of v is the same. By above lemma, we get:

Lemma 3.3. For v a node, assume that we increase (or decrease) the variable $b_{v,t+1}$ at rate h . If the increasing (or decreasing) rate of each $w \in C(v)$ is the same, then in order to keep the **Node Identity Property**, we should set the increasing (or decreasing) rate for each child $w \in C(v)$ as follows:

$$\frac{db_{w,t+1}}{dh} = \frac{1}{\sigma} \cdot \frac{db_{v,t+1}}{dh}$$

Repeatedly applying this lemma, we get the following corollary.

Corollary 3.4. For a node v with $\ell(v) = j$ and a path P from leaf $p_i \in T_v$ to v , if $b_{v,t+1}$ is increased (or decreased) at rate h and the increasing (decreasing) rate of all children of any $v' \in P$ is the same, then $\sum_{v' \in P} \frac{db_{v',t+1}}{dh} = \frac{db_{v,t+1}}{dh} \cdot \psi(j)$, where $\psi(j) = (1 + \frac{1}{\sigma} + \frac{1}{\sigma^2} + \dots + \frac{1}{\sigma^{\ell-j}}) = (1 + \Theta(\frac{1}{\sigma}))$.

We still require the following special case of lemma 3.2. Let w_1 be the first child of the node v . Assume that $b_{w_1,t+1}$ is increased (or decreased) at some rate and the rate of increasing (or decreasing) $b_{w',t+1}$ is the same for every $w' \in C(v)$, $w' \neq w_1$. If $b_{v,t+1}$ is unchanged, then the following claim should hold.

Lemma 3.5. Let w_1, \dots, w_m be the children of a node v . Assume that we increase (or decrease) w_1 at rate h and also increase w_2 to w_m at the same rate h . For $i \geq 2$, let $\frac{db_{w',t+1}}{dh}$ be $\frac{db_{w_i,t+1}}{dh}$. If we would like to maintain the amount $u_{v,t}$ unchanged, then we should have:

$$\frac{db_{w',t+1}}{dh} = \frac{u_{w_1,t} + \frac{|T(w_1)|}{k}}{u_{v,t} + \frac{|T(v)|}{k}} \cdot \left(-\frac{db_{w_1,t+1}}{dh} + \frac{db_{w',t+1}}{dh} \right)$$

Theorem 3.6. When request p_t arrives at time t , in order to keep the dual constraints tight and node identity property, if $a_{S,t}$ is increased with rate 1, we should decrease every $b_{A(p_t,j),t+1}$ ($1 \leq j \leq \ell$) with rate:

$$\frac{db_{A(p_t,j),t+1}}{da_{S,t}} = \frac{2 + \frac{1}{n-1}}{\psi(j)} \left[\left(u_{A(p_t,j),t} + \frac{|T_{A(p_t,j)}|}{k} \right)^{-1} - \left(u_{A(p_t,j-1),t} + \frac{|T_{A(p_t,j-1)}|}{k} \right)^{-1} \right].$$

For each sibling w of $A(p_t, j)$, increase $b_{w,t+1}$ with the following rate:

$$\frac{db_{w,t+1}}{da_{S,t}} = \frac{2 + \frac{1}{n-1}}{\psi(j)} \left(u_{A(p_t,j-1),t} + \frac{|T_{A(p_t,j-1)}|}{k} \right)^{-1}$$

Thus, we design an online algorithm for the fractional k -sever problem as follows (see Algorithm 3.1).

- 1: At time $t = 0$, we set $b_{p,1} = \gamma_p = 0$ for all p and set $b_{A(p,j),1} = 0$ for any $1 \leq j \leq \ell$.
- 2: At time $t \geq 1$, when a request p_t arrives:
- 3: Initially, we set $u_{p,t} = u_{p,t-1}$ for all p , and $b_{p,t+1}$ is initialized to $b_{p,t}$.
- 4: If $u_{p_t,t} = 0$, then do nothing.
- 5: Otherwise, do the following:
- 6: Let $S = \{p : u_{p,t} < 1\}$. Since $k = n - 1$, $|S| > \sum_{x \in S} x_{p,t} = k = n - 1$. So, $S = [n]$.
- 7: While $u_{p_t,t} \neq 0$:
- 8: Increasing $a_{S,t}$ with rate 1;
- 9: For each $1 \leq j \leq \ell$, decrease every $b_{A(p_t,j),t+1}$ with rate:
- 10: $\frac{db_{A(p_t,j),t+1}}{da_{S,t}} = \frac{2 + \frac{1}{n-1}}{\psi(j)} \left[\left(u_{A(p_t,j),t} + \frac{|T_{A(p_t,j)}|}{k} \right)^{-1} - \left(u_{A(p_t,j-1),t} + \frac{|T_{A(p_t,j-1)}|}{k} \right)^{-1} \right],$
- 11: For each sibling w of $A(p_t, j)$, increase $b_{w,t+1}$ with the following rate:
- 12: $\frac{db_{w,t+1}}{da_{S,t}} = \frac{2 + \frac{1}{n-1}}{\psi(j)} \left(u_{A(p_t,j-1),t} + \frac{|T_{A(p_t,j-1)}|}{k} \right)^{-1}$
- 13: For any node v' in the path from w to a leaf in T_w , if w' be the child of v' , $\frac{db_{w',t+1}}{dh} =$
 $\frac{1}{\sigma} \cdot \frac{db_{v',t+1}}{dh}$

Algorithm 3.1: The online primal-dual algorithm for the fractional k -sever problem on a σ -HST.

Theorem 3.7. The online algorithm for the fractional k -sever problem on a σ -HST is of competitive ratio $15 \ln^2(1 + k)$.

In [32], Duru Türkoğlu study the relationship between fractional version and randomized version of the k -sever problem, which is given as follows.

Lemma 3.8. *The fractional k -sever problem is equivalent to the randomized k -sever problem on the line or circle, or if $k = 2$ or $k = n - 1$ for arbitrary metric spaces.*

Thus, we get the following conclusion:

Theorem 3.9. *There is a randomized algorithm with competitive ratio $15 \ln^2(1+k)$ for the k -sever problem on a σ -HST when $n = k + 1$.*

By lemma 2.5, we get the following conclusion:

Theorem 3.10. *There is an $O(\log^2 k \log n)$ competitive randomized algorithm for the k -sever problem on any metric space when $n = k + 1$.*

4 An $O(\ell \log k)$ -competitive Fractional Algorithm for the k -Sever Problem on a Weighted HST with Depth ℓ

In this section, we first give an $O(\ell \log k)$ -competitive fractional algorithm for the k -Sever problem on a weighted σ -HST with depth ℓ .

We give another some notations for a weighted HST. Let \tilde{T} be a weighted σ -HST. For a node whose depth is j , let $\sigma_j = \frac{D(v)}{D(w)} = \frac{d(p(v), v)}{d(v, w)}$ where w is a child of v . By the definition of a weighted σ -HST, $\sigma_j \geq \sigma$ for all $1 \leq j \leq \ell - 1$. For a node $v \in \tilde{T}$, if any leaf $p \in L(T(v))$ such that $u_{p,t} = 1$, we call it a full node. By this definition, for a full node, $u_{v,t} = |L(T_v)|$. Otherwise, we call it non-full node. Let $NFC(v)$ is the set of non-full children node of v , i.e., $NFC(v) = \{w | w \in C(v) \text{ and } w \text{ is a non-full node}\}$. For a node v , let $NL(T_v)$ denote the set of non-full leaf nodes in \tilde{T}_v . Let $S = \{p | u_{p,t} < 1\}$. Let P' denotes the path from p_t to root r : $\{A(p_t, \ell) = P_t, A(p_t, \ell - 1), \dots, A(p_t, 1), \dots, A(p_t, 0) = r\}$. For a node $v \in P'$, if there exists a $p \in S \setminus \{p_t\}$ such that v is the first common ancestor of p_t and p , we call it a common ancestor node in P' . Let $CA(p_t, S)$ denote the set of common ancestor nodes in P' . Suppose that $CA(p_t, S) = \{A(p_t, \ell_h), \dots, A(p_t, \ell_2), A(p_t, \ell_1)\}$, where $\ell_1 < \ell_2 < \dots < \ell_h$. For a node v , let $u_{v,t} = \sum_{p \in S} u_{p,t}$. It is easy to know that $u_{v,t} = \sum_{w \in NFC(v)} u_{w,t}$. Thus, for a full node v , $u_{v,t} = 0$. For any $\ell_h < j < \ell$, $u_{A(p_t, j), t} = u_{A(p_t, \ell), t}$. For any $0 \leq j < \ell_1$, $u_{A(p_t, 0), t} = u_{A(p_t, j), t} = u_{A(p_t, \ell_1), t}$. For any $\ell_{i-1} < j < \ell_i$, $u_{A(p_t, j), t} = u_{A(p_t, \ell_i), t}$.

The primal-dual formulation for the fractional k -sever problem on a weighted HST is the same as that on a HST in section 3. Based on the primal-dual formulation, the design idea of our online algorithm is similar to the design idea in section 3. During the execution of our algorithm, it keeps the following relation between the primal variable $u_{v,t}$ and dual variable $b_{v,t+1}$: $u_{v,t} = f(b_{v,t+1}) = \frac{|NL(T_v)|}{k} (\exp(\frac{b_{v,t+1}}{2D(v)} \ln(1+k)) - 1)$. This relation determines how much mass of $u_{p_t, t}$ should be gradually moved out of leaf p_t and how it should be distributed among other leaves $S \setminus \{p_t\}$ until p_t is completely fetched into the cache, i.e. $u_{p_t, t} = 0$. Thus, at any time t , the algorithm maintains a distribution $(u_{p_1, t}, \dots, u_{p_n, t})$ on the leaves such that $\sum_{p \in [n]} u_{p, t} = n - k$.

In order to compute the the exact rate at which we should move mass $u_{p_t, t}$ from p_t through its ancestor nodes at time t to other leaves $S \setminus \{p_t\}$ in the weighted σ -HST, using similar argument to that in section 3, we get following several claims. Because of the space limit, we put their proofs in the Appendix.

Lemma 4.1. $\frac{du_{v,t}}{db_{v,t+1}} = \frac{\ln(1+k)}{2D(v)}(u_{v,t} + \frac{|NL(T_v)|}{k}).$

Proof. Since $u_{v,t} = \frac{|NL(T_v)|}{k}(\exp(\frac{b_{v,t+1}}{2D(v)} \ln(1+k)) - 1)$, we take the derivative over $b_{v,t+1}$ and get the claim. \square

Lemma 4.2. For a node v with $\ell(v) = j$, if we increase variable $b_{v,t+1}$ at rate h , then we have the following equality:

$$\frac{1}{\sigma_j}(u_{v,t} + \frac{|NL(T_v)|}{k}) \cdot \frac{db_{v,t+1}}{dh} = \sum_{w \in NFC(v)} \frac{db_{w,t+1}}{dh} \cdot (u_{w,t} + \frac{|NL(T_w)|}{k})$$

Lemma 4.3. For v a node with $\ell(v) = j$, assume that we increase (or decrease) the variable $b_{v,t+1}$ at rate h . If the increasing (or decreasing) rate of each $w \in NFC(v)$ is the same, then in order to keep the **Node Identity Property**, we should set the increasing (or decreasing) rate for each child $w \in NFC(v)$ as follows:

$$\frac{db_{w,t+1}}{dh} = \frac{1}{\sigma_j} \cdot \frac{db_{v,t+1}}{dh}$$

Repeatedly applying this lemma, we get the following corollary.

Corollary 4.4. For a node v with $\ell(v) = j$ and a path P from leaf $p_i \in T_v$ to v , if $b_{v,t+1}$ is increased (or decreased) at rate h and the increasing (decreasing) rate of all children of any $v' \in P$ is the same, then $\sum_{v' \in P} \frac{db_{v',t+1}}{dh} = \frac{db_{v,t+1}}{dh} \cdot \phi(j)$, where $\phi(j) = (1 + \frac{1}{\sigma_j} + \frac{1}{\sigma_j \cdot \sigma_{j+1}} + \dots + \frac{1}{\sigma_j \cdot \sigma_{j+1} \dots \sigma_{\ell-1}}) \leq (1 + \Theta(\frac{1}{\sigma}))$.

Lemma 4.5. Let w_1, \dots, w_m be the non-full children node of a node v (i.e., any $w_i \in NFC(v)$). Assume that we increase (or decrease) w_1 at rate h and also increase w_2 to w_m at the same rate h . For $i \geq 2$, let $\frac{db_{w',t+1}}{dh}$ be $\frac{db_{w_i,t+1}}{dh}$. If we would like to maintain the amount $u_{v,t}$ unchanged, then we should have:

$$\frac{db_{w',t+1}}{dh} = \frac{u_{w_1,t} + \frac{|NL(T_{w_1})|}{k}}{u_{v,t} + \frac{|NL(T_v)|}{k}} \cdot (-\frac{db_{w_1,t+1}}{dh} + \frac{db_{w',t+1}}{dh})$$

Theorem 4.6. When request p_t arrives at time t , in order to keep the dual constraints tight and node identity property, if $a_{S,t}$ is increased with rate 1, we should decrease every $b_{A(p_t,j),t+1}$ for each $j \in \{\ell_1 + 1, \ell_2 + 1, \dots, \ell_h + 1\}$ with rate:

$$\frac{db_{A(p_t,j),t+1}}{da_{S,t}} = \frac{u_{r,t} + \frac{|S|}{k}}{\phi(j)} [(u_{A(p_t,j),t} + \frac{|NL(T_{A(p_t,j)})|}{k})^{-1} - (u_{A(p_t,j-1),t} + \frac{|NL(T_{A(p_t,j-1)})|}{k})^{-1}].$$

For each sibling $w \in NFC(v)$ of $A(p_t, j)$, increase $b_{w,t+1}$ with the following rate:

$$\frac{db_{w,t+1}}{da_{S,t}} = \frac{u_{r,t} + \frac{|S|}{k}}{\phi(j)} (u_{A(p_t,j-1),t} + \frac{|NL(T_{A(p_t,j-1)})|}{k})^{-1}$$

Thus, we design an online algorithm for the fractional k -sever problem on a weighted σ -HST as follows (see Algorithm 4.1).

Theorem 4.7. The online algorithm for the fractional k -sever problem on a weighted σ -HST with depth ℓ is of competitive ratio $4\ell \ln(1+k)$.

By lemma 2.7, we get:

Theorem 4.8. There exists an $O(\log k \log n)$ -competitive fractional algorithm for the k -sever problem on any σ -HST.

In [3], Nikhil Bansal et al. show the following conclusion.

- 1: At time $t = 0$, we set $b_{p,1} = \gamma_p = 0$ for all p .
- 2: At time $t \geq 1$, when a request p_t arrives:
- 3: Initially, we set $u_{p,t} = u_{p,t-1}$ for all p , and $b_{p,t+1}$ is initialized to $b_{p,t}$.
- 4: If $u_{p_t,t} = 0$, then do nothing.
- 5: Otherwise, do the following:
- 6: Let $S = \{p : u_{p,t} < 1\}$. Suppose that $CA(p_t, S) = \{A(p_t, \ell_h), \dots, A(p_t, \ell_2), A(p_t, \ell_1)\}$, where $0 \leq \ell_1 < \ell_2 < \dots < \ell_h < \ell$.
- 7: While $u_{p_t,t} \neq 0$:
- 8: Increasing $a_{S,t}$ with rate 1;
- 9: For each $j \in \{\ell_1 + 1, \ell_2 + 1, \dots, \ell_h + 1\}$, decrease every $b_{A(p_t,j),t+1}$ with rate:
- 10: $\frac{db_{A(p_t,j),t+1}}{da_{S,t}} = \frac{u_{r,t} + \frac{|S|}{k}}{\phi(j)} \left[(u_{A(p_t,j),t} + \frac{|NL(T_{A(p_t,j)})|}{k})^{-1} - (u_{A(p_t,j-1),t} + \frac{|NL(T_{A(p_t,j-1)})|}{k})^{-1} \right]$.
- 11: For each sibling $w \in NFC(v)$ of $A(p_t, j)$, increase $b_{w,t+1}$ with the following rate:
- 12: $\frac{db_{w,t+1}}{da_{S,t}} = \frac{u_{r,t} + \frac{|S|}{k}}{\phi(j)} (u_{A(p_t,j-1),t} + \frac{|NL(T_{A(p_t,j-1)})|}{k})^{-1}$
- 13: For any node v' in the path from w to a leaf in $NL(T_w)$, if $w' \in NFC(v')$ and $\ell(v') = j$, $\frac{db_{w',t+1}}{dh} = \frac{1}{\sigma_j} \cdot \frac{db_{v',t+1}}{dh}$
- 14: For $p \in S \setminus \{p_t\}$, if some $u_{p,t}$ reaches the value of 1, then we update $S \leftarrow S \setminus \{p\}$ and the set $NFC(v)$ for each ancestor node v of p .
- 15:
- 16:

Algorithm 4.1: The online primal-dual algorithm for the fractional k -sever problem on a weighted σ -HST.

Lemma 4.9. *Let T be a σ -HST with $\sigma > 5$. Then any online fractional k -sever algorithm on T can be converted into a randomized k -sever algorithm on T with an $O(1)$ factor loss in the competitive ratio.*

Thus, we get the following conclusion by Theorem 4.8:

Theorem 4.10. *Let T be a σ -HST with $\sigma > 5$. There is a randomized algorithm for the k -sever problem with a competitive ratio of $O(\log k \log n)$ on T .*

By lemma 2.5, we get the following conclusion:

Theorem 4.11. *For any metric space, there is a randomized algorithm for the k -sever problem with a competitive ratio of $O(\log k \log^2 n)$.*

5 Conclusion

In this paper, for any metric space with n points, we show that there exist a randomized algorithm with $O(\log k \log^2 n)$ -competitive ratio for the k -sever problem, which improved the previous best competitive ratio $O(\log^2 k \log^3 n \log \log n)$.

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Appendix

Proofs for claims in section 3

The proof for Lemma 3.2 is as follows:

Proof. Since it is required to maintain $u_{v,t} = \sum_{w \in C(v)} u_{w,t}$, we take the derivative of both sides and get that:

$$\frac{du_{v,t}}{db_{v,t+1}} \cdot \frac{db_{v,t+1}}{dh} = \sum_{w \in C(v)} \frac{du_{w,t}}{db_{w,t+1}} \cdot \frac{db_{w,t+1}}{dh}.$$

By lemma 3.1, we get: $\frac{\ln(1+k)}{2D(v)}(u_{v,t} + \frac{|T_v|}{k}) = \sum_{w \in C(v)} \frac{db_{w,t+1}}{dh} \cdot \frac{\ln(1+k)}{2D(w)}(u_{w,t} + \frac{|T_w|}{k})$.

Since $\frac{D(v)}{D(w)} = \sigma$, we get :

$$\frac{1}{\sigma}(u_{v,t} + \frac{|T_v|}{k}) \cdot \frac{db_{v,t+1}}{dh} = \sum_{w \in C(v)} \frac{db_{w,t+1}}{dh} \cdot (u_{w,t} + \frac{|T_w|}{k}) \quad \square$$

The proof for Lemma 3.3 is as follows:

Proof. By above Lemma 3.2, if the increasing (or decreasing) rate of each $w \in C(v)$ is the same, we get that:

$$\frac{1}{\sigma}(u_{v,t} + \frac{|T_v|}{k}) \cdot \frac{db_{v,t+1}}{dh} = \frac{db_{w,t+1}}{dh} \cdot \sum_{w \in C(v)} (u_{w,t} + \frac{|T_w|}{k}) = \frac{db_{w,t+1}}{dh} \cdot (u_{v,t} + \frac{|T_v|}{k}).$$

So, we get that: $\frac{db_{w,t+1}}{dh} = \frac{1}{\sigma} \cdot \frac{db_{v,t+1}}{dh}$ □

The proof for Lemma 3.5 is as follows:

Proof. By lemma 3.2, in order to keep the amount $u_{v,t}$ unchanged, we get:

$$\frac{db_{w_1,t+1}}{dh} \cdot (u_{w_1,t} + \frac{|T(w_1)|}{k}) + \sum_{w \in C(v) \setminus \{w_1\}} \frac{db_{w,t+1}}{dh} \cdot (u_{w,t} + \frac{|T(w)|}{k}) = 0.$$

$$\text{Thus, } (\frac{db_{w_1,t+1}}{dh} - \frac{db_{w',t+1}}{dh}) \cdot (u_{w_1,t} + \frac{|T(w_1)|}{k}) + \frac{db_{w',t+1}}{dh} \cdot \sum_{w \in C(v)} (u_{w,t} + \frac{|T(w)|}{k}) = 0$$

$$\text{So, } (\frac{db_{w_1,t+1}}{dh} - \frac{db_{w',t+1}}{dh}) \cdot (u_{w_1,t} + \frac{|T(w_1)|}{k}) + \frac{db_{w',t+1}}{dh} \cdot (u_{v,t} + \frac{|T(v)|}{k}) = 0$$

Hence, we get the claim.

□

The proof for Theorem 3.6 is as follows:

Proof. When request p_t arrives at time t , we move mass $u_{p_t,t}$ from p_t through its ancestor nodes to other leaves $[n] \setminus \{p_t\}$, i.e. $u_{p_t,t}$ is decreased and $u_{p,t}$ is increased for any $p \in [n] \setminus \{p_t\}$. Since these mass moves out of each subtree $T_{A(p_t,j)}$ for each $1 \leq j \leq \ell$, $u_{A(p_t,j),t}$ is decreased. By $u_{A(p_t,j),t} = f(b_{A(p_t,j),t+1}) = \frac{|T_v|}{k}(\exp(\frac{b_{A(p_t,j),t+1}}{2D(v)} \ln(1+k)) - 1)$ (we need to keep this relation during the algorithm), $b_{A(p_t,j),t+1}$ also decreases for each $1 \leq j \leq \ell$. On the other hand, $u_{p,t}$ is increased for each $p \in [n] \setminus \{p_t\}$. Thus, for each node v whose T_v doesn't contain p_t , its mass $u_{v,t}$ is also increased. For each node v whose T_v doesn't contain p_t , it must be a sibling of some node $A(p_t, j)$. For each $1 \leq j \leq \ell$, we assume that all siblings v' of node $A(p_t, j)$ increase $b_{v',t+1}$ at the same rate.

In the following, we will compute the increasing (or decreasing) rate of all dual variables in the σ -HST regarding a_S . For $1 \leq j \leq \ell$, let $\nabla b_j = -\frac{b_{A(p_t,j),t+1}}{da_S}$ be the decreasing rate of $b_{A(p_t,j),t+1}$ regarding a_S . For $1 \leq j \leq \ell$, let $\nabla b'_j = \frac{b_{w,t+1}}{da_S}$ be the increasing rate of $b_{w,t+1}$ for any siblings w of $A(p_t, j)$ regarding a_S .

Using from top to down method, we can get a set of equations about the quantities ∇b_j and $\nabla b'_j$. First, we consider the siblings of $A(p_t, 1)$ (i.e. those nodes are children of root r , but they are not $A(p_t, 1)$). Let w be one of these siblings. If $b_{w,t+1}$ is raised by $\nabla b'_1$, by Corollary 3.4, the sum of $\nabla b'$ on any path from a leaf in T_w to w must be $\psi(1) \cdot \nabla b'_1$. Since a_S is increasing with rate 1, it forces $\psi(1) \cdot \nabla b'_1 = 1$ in order to maintain the dual constraint (3) tight for leaves in T_w .

This considers the dual constraints for these leaves. Now, this increasing mass must be canceled out by decreasing the mass in $T_{A(p_t,1)}$ since the mass $u_{r,t}$ in T_r is not changed. Thus, in order to maintain the “Node Identity Property” of root, by Lemma 3.5, we must set ∇b_1 such that:

$$\nabla b'_1 = (\nabla b_1 + \nabla b'_1) \cdot \frac{(u_{A(p_t,1),t} + \frac{|T_{A(p_t,1)}|}{k})}{1 + \frac{n}{n-1}}$$

For siblings of node $A(p_t, 2)$, we use the similar argument. Let w be a sibling of $A(p_t, 2)$. Consider a path from a leaf in T_w to the w . Their dual constraint (3) already grows at rate $1 + \psi(1)\nabla b_1$. This must be canceled out by increasing $b_{w,t+1}$, and if $b_{w,t+1}$ is raised by $\nabla b'_2$, by corollary 3.4, the sum of $\nabla b'$ on any path from a leaf in T_w to w must be $\psi(2) \cdot \nabla b'_2$. Thus, $\nabla b'_2$ must be set such that: $\psi(2) \cdot \nabla b'_2 = 1 + \psi(1) \cdot \nabla b_1$

Again, this increasing mass must be canceled out by decreasing the mass in $T_{A(p_t,2)}$. In order to keep the “Node Identity Property” of $A(p_t, 1)$, by Lemma 3.5, we must set ∇b_2 such that:

$$\nabla b'_2 = (\nabla b_2 + \nabla b'_2) \cdot \frac{(u_{A(p_t,2),t} + \frac{|T_{A(p_t,2)}|}{k})}{(u_{A(p_t,1),t} + \frac{|T_{A(p_t,1)}|}{k})}$$

Continuing this method, we obtain a system of linear equations about all ∇b_j and $\nabla b'_j$ ($1 \leq j \leq \ell$). For maintaining the dual constraints tight, we get the following equations:

$$\begin{aligned} \psi(1) \cdot \nabla b'_1 &= 1 \\ \psi(2) \cdot \nabla b'_2 &= 1 + \psi(1) \cdot \nabla b_1 \\ &\vdots \\ \psi(\ell) \cdot \nabla b'_\ell &= 1 + \sum_{i=1}^{\ell-1} \psi(i) \nabla b_i \end{aligned}$$

For keeping the node identity property, we get the following equations:

$$\nabla b'_1 = (\nabla b_1 + \nabla b'_1) \cdot \frac{(u_{A(p_t,1),t} + \frac{|T_{A(p_t,1)}|}{k})}{1 + \frac{n}{n-1}}$$

$$\begin{aligned} \nabla b'_2 &= (\nabla b_2 + \nabla b'_2) \cdot \frac{(u_{A(p_t,2),t} + \frac{|T(A(p_t,2))|}{k})}{(u_{A(p_t,1),t} + \frac{|T(A(p_t,1))|}{k})} \\ &\vdots \\ \nabla b'_\ell &= (\nabla b_\ell + \nabla b'_\ell) \cdot \frac{(u_{A(p_t,\ell),t} + \frac{|T(A(p_t,\ell))|}{k})}{(u_{A(p_t,\ell-1),t} + \frac{|T(A(p_t,\ell-1))|}{k})}. \end{aligned}$$

We continue to solve the system of linear equations.

For each $1 \leq j \leq \ell$,

$$\begin{aligned} \psi(j) \cdot \nabla b'_j &= 1 + \sum_{i=1}^{j-1} \psi(i) \nabla b_i \\ &= 1 + \sum_{i=1}^{j-2} \psi(i) \nabla b_i + \psi(j-1) \nabla b_{j-1} \\ &= \psi(j-1) \cdot \nabla b'_{j-1} + \psi(j-1) \nabla b_{j-1} \\ &= \psi(j-1) \cdot (\nabla b'_{j-1} + \nabla b_{j-1}). \end{aligned}$$

Since $\nabla b'_{j-1} = (\nabla b_{j-1} + \nabla b'_{j-1}) \cdot \frac{(u_{A(p_t,j-1),t} + \frac{|T(A(p_t,j-1))|}{k})}{(u_{A(p_t,j-2),t} + \frac{|T(A(p_t,j-2))|}{k})}$, we get:

$$\psi(j) \nabla b'_j = \psi(j-1) \nabla b'_{j-1} \cdot \frac{(u_{A(p_t,j-2),t} + \frac{|T(A(p_t,j-2))|}{k})}{(u_{A(p_t,j-1),t} + \frac{|T(A(p_t,j-1))|}{k})}$$

Solving the recursion, we get:

$$\begin{aligned} \nabla b'_j &= \frac{1 + \frac{n}{n-1}}{\psi(j) \cdot (u_{A(p_t,j-1),t} + \frac{|T(A(p_t,j-1))|}{k})} \\ \nabla b_j &= \frac{1 + \frac{n}{n-1}}{\psi(j)} \cdot [(u_{A(p_t,j),t} + \frac{|T(A(p_t,j))|}{k})^{-1} - (u_{A(p_t,j-1),t} + \frac{|T(A(p_t,j-1))|}{k})^{-1}] \end{aligned}$$

□

The proof for Theorem 3.7 is as follows:

Proof. Let P denote the value of the objective function of the primal solution and D denote the value of the objective function of the dual solution. Initially, let $P = 0$ and $D = 0$. In the following, we prove three claims:

- (1) The primal solution produced by the algorithm is feasible.
- (2) The dual solution produced by the algorithm is feasible.
- (3) $P \leq 15 \ln^2(1+k)D$.

By three claims and weak duality of linear programs, the theorem follows immediately.

First, we prove the claim (1) as follows. At any time t , since $S = [n]$ and the algorithm keeps $\sum_{p \in [n]} = n - k = 1 = |S| - k$. So, the primal constraints (3.1) are satisfied.

Second, we prove the claim (2) as follows. By theorem 3.6, the dual constraints (3.6) are satisfied. Obviously, dual constraints (3.7) are satisfied. For any node v , if $b_{v,t+1} = 0$, then $u_{v,t} = 0$; if $b_{v,t+1} = 2D(v)$, then $u_{v,t} = |T_v|$. Thus, the dual constraints (3.8) are satisfied.

Third, we prove claim (3) as follows. If the algorithm increases the variables $a_{S,t}$ at some time t , then: $\frac{\partial D}{\partial a_{S,t}} = |S| - k = n - (n-1) = 1$. Let's compute the primal cost. At depth j ($1 \leq j \leq \ell$), we compute the movement cost of our algorithm by the change of $\nabla b'_j$ as follows.

$$\begin{aligned} &\sum_{w \in C(A(p_t,j-1)) \setminus \{A(p_t,j)\}} 2D(w) \cdot \frac{du_{w,t}}{db'_j} \cdot \frac{db'_j}{a_S} \\ &= \nabla b'_j \cdot \sum_{w \in C(A(p_t,j-1)) \setminus \{A(p_t,j)\}} 2D(w) \cdot \frac{\ln(1+k)}{2D(w)} (u_{w,t} + \frac{|T_w|}{k}) \end{aligned}$$

$$\begin{aligned}
&= \frac{2+\frac{1}{n-1}}{\psi(j)} \cdot \ln(1+k) \cdot \frac{\sum_{w \in C(A(p_t, j-1)) \setminus \{A(p_t, j)\}} (u_{w,t} + \frac{|T_w|}{k})}{u_{A(p_t, j-1), t} + \frac{|T_{A(p_t, j-1)}|}{k}} \\
&\leq \frac{5}{2} \cdot \ln(1+k) \cdot \frac{\sum_{w \in C(A(p_t, j-1)) \setminus \{A(p_t, j)\}} (u_{w,t} + \frac{|T_w|}{k})}{u_{A(p_t, j-1), t} + \frac{|T_{A(p_t, j-1)}|}{k}}
\end{aligned}$$

Let B_j denote $u_{A(p_t, j), t} + \frac{|T_{A(p_t, j)}|}{k}$. Then $\sum_{w \in C(A(p_t, j-1)) \setminus \{A(p_t, j)\}} (u_{w,t} + \frac{|T_w|}{k}) = B_{j-1} - B_j$.

Hence, the total movement cost over all ℓ levels is

$$\begin{aligned}
&\frac{5}{2} \ln(1+k) \cdot \sum_{j=1}^{\ell} \frac{B_{j-1} - B_j}{B_{j-1}} \\
&= \frac{5}{2} \ln(1+k) \cdot \sum_{j=1}^{\ell} (1 - \frac{B_j}{B_{j-1}}) \\
&\leq \frac{5}{2} \ln(1+k) \cdot \sum_{j=1}^{\ell} \ln \frac{B_j}{B_{j-1}} \\
&= \frac{5}{2} \ln(1+k) \cdot \ln \frac{B_0}{B_{\ell}} \\
&= \frac{5}{2} \ln(1+k) \cdot \ln \frac{B_0}{u_{p_t, t} + \frac{1}{k}} \\
&\leq \frac{5}{2} \ln(1+k) \cdot \ln k B_0 \\
&\leq \frac{5}{2} \ln(1+k) \cdot 2 \ln k \cdot B_0 \\
&\leq \frac{5}{2} \ln(1+k) \cdot 2 \ln k \cdot (2 + \frac{1}{n-1}) \\
&= 15 \ln^2(1+k)
\end{aligned}$$

where the first inequality holds since $1 \leq y - \ln y$ for any $0 \leq y \leq 1$.

Thus, we get $P \leq 15 \ln^2(1+k)D$.

Let OPT be the cost of the best offline algorithm. P_{min} be the optimal primal solution and D_{max} be the optimal dual solution. Then, $P_{min} \leq OPT$ since OPT is a feasible solution for the primal program. Based on the weak duality, $D_{max} \leq P_{min}$. Hence, $\frac{P}{OPT} \leq \frac{P}{P_{min}} \leq \frac{15 \ln^2(1+k)D}{P_{min}} \leq \frac{15 \ln^2(1+k)D_{max}}{P_{min}} \leq \frac{15 \ln^2(1+k)P_{min}}{P_{min}} = 15 \ln^2(1+k)$.

So, the competitive ratio of this algorithm is $15 \ln^2(1+k)$. □

Proofs for claims in section 4

The proof for Lemma 4.2 is as follows:

Proof. Since it is required to maintain $u_{v,t} = \sum_{w \in NFC(v)} u_{w,t}$, we take the derivative of both sides

and get that:

$$\frac{du_{v,t}}{db_{v,t+1}} \cdot \frac{db_{v,t+1}}{dh} = \sum_{w \in NFC(v)} \frac{du_{w,t}}{db_{w,t+1}} \cdot \frac{db_{w,t+1}}{dh}.$$

By lemma 3.1, we get: $\frac{\ln(1+k)}{2D(v)}(u_{v,t} + \frac{|NL(T_v)|}{k}) = \sum_{w \in NFC(v)} \frac{db_{w,t+1}}{dh} \cdot \frac{\ln(1+k)}{2D(w)}(u_{w,t} + \frac{|NL(T_w)|}{k})$.

Since $\frac{D(v)}{D(w)} = \sigma_j$, we get :

$$\frac{1}{\sigma_j}(u_{v,t} + \frac{|NL(T_v)|}{k}) \cdot \frac{db_{v,t+1}}{dh} = \sum_{w \in NFC(v)} \frac{db_{w,t+1}}{dh} \cdot (u_{w,t} + \frac{|NL(T_w)|}{k})$$

□

The proof for Lemma 4.3 is as follows:

Proof. By Lemma 4.2, if the increasing (or decreasing) rate of each $w \in NFC(v)$ is the same, we get that:

$$\frac{1}{\sigma_j}(u_{v,t} + \frac{|NL(T_v)|}{k}) \cdot \frac{db_{v,t+1}}{dh} = \frac{db_{w,t+1}}{dh} \cdot \sum_{w \in NFC(v)} (u_{w,t} + \frac{|NL(T_w)|}{k}) = \frac{db_{w,t+1}}{dh} \cdot (u_{v,t} + \frac{|NL(T_v)|}{k}).$$

So, we get that: $\frac{db_{w,t+1}}{dh} = \frac{1}{\sigma_j} \cdot \frac{db_{v,t+1}}{dh}$ □

The proof for Lemma 4.5 is as follows:

Proof. By lemma 4.2, in order to keep the amount $u_{v,t}$ unchanged, we get:

$$\frac{db_{w_1,t+1}}{dh} \cdot (u_{w_1,t} + \frac{|NL(T_{w_1})|}{k}) + \sum_{w \in NFC(v) \setminus \{w_1\}} \frac{db_{w,t+1}}{dh} \cdot (u_{w,t} + \frac{|NL(T_w)|}{k}) = 0.$$

$$\text{Thus, } (\frac{db_{w_1,t+1}}{dh} - \frac{db_{w',t+1}}{dh}) \cdot (u_{w_1,t} + \frac{|NL(T_{w_1})|}{k}) + \frac{db_{w',t+1}}{dh} \cdot \sum_{w \in NFC(v)} (u_{w,t} + \frac{|NL(T_w)|}{k}) = 0$$

$$\text{So, } (\frac{db_{w_1,t+1}}{dh} - \frac{db_{w',t+1}}{dh}) \cdot (u_{w_1,t} + \frac{|NL(T_{w_1})|}{k}) + \frac{db_{w',t+1}}{dh} \cdot (u_{v,t} + \frac{|NL(T_v)|}{k}) = 0$$

Hence, we get the claim. □

The proof for Theorem 4.6 is as follows:

Proof. When request p_t arrives at time t , we move mass $u_{p_t,t}$ from p_t through its ancestor nodes to other non-full leaves nodes $S \setminus \{p_t\}$, i.e. $u_{p_t,t}$ is decreased and $u_{p,t}$ is increased for any $p \in S \setminus \{p_t\}$. Since these mass moves out of each subtree $T_{A(p_t,j)}$ for each $1 \leq j \leq \ell$, $u_{A(p_t,j),t}$ is decreased. By $u_{A(p_t,j),t} = f(b_{A(p_t,j),t+1}) = \frac{|NL(T_v)|}{k} (\exp(\frac{b_{A(p_t,j),t+1}}{2D(v)} \ln(1+k)) - 1)$ (we need to keep this relation during the algorithm), $b_{A(p_t,j),t+1}$ also decreases for each $1 \leq j \leq \ell$. On the other hand, $u_{p,t}$ is increased for each $p \in S \setminus \{p_t\}$. Thus, for each non-full node v whose T_v doesn't contain p_t , its mass $u_{v,t}$ is also increased. For each non-full node v whose T_v doesn't contain p_t , it must be a sibling of some node $A(p_t,j)$ where $j \in \{\ell_1, \dots, \ell_h\}$. We assume that all siblings v' of any node v increase $b_{v',t+1}$ at the same rate

In the following, we will compute the increasing (or decreasing) rate of all dual variables in the weighted σ -HST regarding a_S . For $1 \leq j \leq \ell$, let $\nabla b_j = -\frac{b_{A(p_t,j),t+1}}{da_S}$ be the decreasing rate of $b_{A(p_t,j),t+1}$ regarding a_S . For each $j \in \{\ell_1 + 1, \ell_2 + 1, \dots, \ell_h + 1\}$, let $\nabla b'_j = \frac{b_{w,t+1}}{da_S}$ be the increasing rate of $b_{w,t+1}$ for any siblings $w \in NFC(A(p_t,j))$ of $A(p_t,j)$ regarding a_S .

Using from top to down method, we can get a set of equations about the quantities ∇b_j and $\nabla b'_j$. First, we consider the siblings of $A(p_t, \ell_1 + 1)$ (i.e. those nodes are children of $A(p_t, \ell_1)$, but they are not $A(p_t, \ell_1 + 1)$). Let w be one of these siblings. If $b_{w,t+1}$ is raised by $\nabla b'_{\ell_1+1}$, by Corollary 4.4, the sum of $\nabla b'$ on any path from a leaf in T_w to w must be $\phi(\ell_1 + 1) \cdot \nabla b'_{\ell_1+1}$. Since a_S is increasing with rate 1, it forces $\phi(\ell_1 + 1) \cdot \nabla b'_{\ell_1+1} = 1$ in order to maintain the dual constraint (3) tight for non-full leaf nodes in T_w .

This considers the dual constraints for these non-full leaf nodes. Now, this increasing mass must be canceled out by decreasing the mass in $T_{A(p_t, \ell_1 + 1)}$ since the mass $u_{A(p_t, \ell_1),t}$ in $T_{A(p_t, \ell_1)}$ is not changed. Thus, in order to maintain the “Node Identity Property” of $A(p_t, \ell_1)$, by Lemma 4.5, we must set ∇b_1 such that:

$$\begin{aligned} \nabla b'_{\ell_1+1} &= (\nabla b_{\ell_1+1} + \nabla b'_{\ell_1+1}) \cdot \frac{(u_{A(p_t, \ell_1+1),t} + \frac{|NL(T_{A(p_t, \ell_1+1)})|}{k})}{u_{A(p_t, \ell_1),t} + \frac{|S|}{k}} \\ &= (\nabla b_{\ell_1+1} + \nabla b'_{\ell_1+1}) \cdot \frac{(u_{A(p_t, \ell_2),t} + \frac{|NL(T_{A(p_t, \ell_2)})|}{k})}{u_r + \frac{|S|}{k}} \end{aligned}$$

For siblings of node $A(p_t, \ell_2 + 1)$, we use the similar argument. Let w be a sibling of $A(p_t, \ell_2 + 1)$. Consider a path from a non-full leaf node in T_w to the w . Their dual constraint (3.6) already grows

at rate $1 + \phi(\ell_1 + 1) \nabla b_{\ell_1+1}$. This must be canceled out by increasing $b_{w,t+1}$, and if $b_{w,t+1}$ is raised by $\nabla b'_{\ell_2+1}$, by corollary 4.4, the sum of $\nabla b'$ on any path from a leaf in T_w to w must be $\phi(2) \cdot \nabla b'_2$. Thus, $\nabla b'_2$ must be set such that: $\phi(\ell_2 + 1) \cdot \nabla b'_{\ell_2+1} = 1 + \phi(\ell_1 + 1) \cdot \nabla b_{\ell_1+1}$.

Again, this increasing mass must be canceled out by decreasing the mass in $T_{A(p_t, \ell_2)}$. In order to keep the “Node Identity Property” of $A(p_t, \ell_2)$, by Lemma 4.5. we must set ∇b_{ℓ_2+1} such that:

$$\begin{aligned} \nabla b'_{\ell_2+1} &= (\nabla b_{\ell_2+1} + \nabla b'_{\ell_2+1}) \cdot \frac{(u_{A(p_t, \ell_2+1), t} + \frac{|NL(T_{A(p_t, \ell_2+1)})|}{k})}{(u_{A(p_t, \ell_2), t} + \frac{|NL(T_{A(p_t, \ell_2)})|}{k})} \\ &= (\nabla b_{\ell_2+1} + \nabla b'_{\ell_2+1}) \cdot \frac{(u_{A(p_t, \ell_3), t} + \frac{|NL(T_{A(p_t, \ell_3)})|}{k})}{(u_{A(p_t, \ell_2), t} + \frac{|NL(T_{A(p_t, \ell_2)})|}{k})} \end{aligned}$$

Continuing this method, we obtain a system of linear equations about all ∇b_j and $\nabla b'_j$ ($j \in \{\ell_1+1, \ell_2+1, \dots, \ell_h+1\}$). For maintaining the dual constraints tight, we get the following equations:

$$\begin{aligned} \phi(\ell_1 + 1) \cdot \nabla b'_{\ell_1+1} &= 1 \\ \phi(\ell_2 + 1) \cdot \nabla b'_{\ell_2+1} &= 1 + \phi(\ell_1 + 1) \cdot \nabla b_{\ell_1+1} \\ &\vdots \\ \psi(\ell_h + 1) \cdot \nabla b'_{\ell_h+1} &= 1 + \sum_{i=1}^{h-1} \phi(\ell_i + 1) \nabla b_{\ell_i+1} \end{aligned}$$

For keeping the node identity property, we get the following equations:

$$\begin{aligned} \nabla b'_{\ell_1+1} &= (\nabla b_{\ell_1+1} + \nabla b'_{\ell_1+1}) \cdot \frac{(u_{A(p_t, \ell_2), t} + \frac{|NL(T_{A(p_t, \ell_2)})|}{k})}{u_r + \frac{|S|}{k}} \\ \nabla b'_{\ell_2+1} &= (\nabla b_{\ell_2+1} + \nabla b'_{\ell_2+1}) \cdot \frac{(u_{A(p_t, \ell_3), t} + \frac{|NL(T_{A(p_t, \ell_3)})|}{k})}{(u_{A(p_t, \ell_2), t} + \frac{|NL(T_{A(p_t, \ell_2)})|}{k})} \\ &\vdots \\ \nabla b'_{\ell_h+1} &= (\nabla b_{\ell_h+1} + \nabla b'_{\ell_h+1}) \cdot \frac{(u_{A(p_t, \ell), t} + \frac{|NL(T_{A(p_t, \ell)})|}{k})}{(u_{A(p_t, \ell_h), t} + \frac{|NL(T_{A(p_t, \ell_h)})|}{k})}. \end{aligned}$$

We continue to solve the system of linear equations.

For each $1 \leq j \leq h$,

$$\begin{aligned} \phi(\ell_j + 1) \cdot \nabla b'_{\ell_j+1} &= 1 + \sum_{i=1}^{j-1} \phi(\ell_i + 1) \nabla b_{\ell_i+1} \\ &= 1 + \sum_{i=1}^{j-2} \phi(\ell_i + 1) \nabla b_{\ell_i+1} + \phi(\ell_{j-1} + 1) \nabla b_{\ell_{j-1}+1} \\ &= \phi(\ell_{j-1} + 1) \cdot \nabla b'_{\ell_{j-1}+1} + \phi(j-1) \nabla b_{\ell_{j-1}+1} \\ &= \phi(\ell_{j-1} + 1) \cdot (\nabla b'_{\ell_{j-1}+1} + \nabla b_{\ell_{j-1}+1}). \end{aligned}$$

Since $\nabla b'_{\ell_{j-1}+1} = (\nabla b_{\ell_{j-1}+1} + \nabla b'_{\ell_{j-1}+1}) \cdot \frac{(u_{A(p_t, \ell_j), t} + \frac{|NL(T_{A(p_t, \ell_j)})|}{k})}{(u_{A(p_t, \ell_{j-1}), t} + \frac{|NL(T_{A(p_t, \ell_{j-1})})|}{k})}$, we get:

$$\psi(\ell_j + 1) \nabla b'_{\ell_j+1} = \psi(\ell_{j-1} + 1) \nabla b'_{\ell_{j-1}+1} \cdot \frac{(u_{A(p_t, \ell_{j-1}), t} + \frac{|NL(T_{A(p_t, \ell_{j-1})})|}{k})}{(u_{A(p_t, \ell_j), t} + \frac{|NL(T_{A(p_t, \ell_j)})|}{k})}$$

Solving the recursion, we get:

$$\begin{aligned} \nabla b'_{\ell_j+1} &= \frac{u_r + \frac{|S|}{k}}{\phi(\ell_j+1) \cdot (u_{A(p_t, \ell_j), t} + \frac{|NL(T_{A(p_t, \ell_j)})|}{k})} \\ \nabla b_{\ell_j+1} &= \frac{u_r + \frac{|S|}{k}}{\phi(\ell_j+1)} \cdot \left[(u_{A(p_t, \ell_{j+1}), t} + \frac{|NL(T_{A(p_t, \ell_{j+1})})|}{k})^{-1} - (u_{A(p_t, \ell_j), t} + \frac{|NL(T_{A(p_t, \ell_j)})|}{k})^{-1} \right] \\ &= \frac{u_r + \frac{|S|}{k}}{\phi(\ell_j+1)} \cdot \left[(u_{A(p_t, \ell_j+1), t} + \frac{|NL(T_{A(p_t, \ell_j+1)})|}{k})^{-1} - (u_{A(p_t, \ell_j), t} + \frac{|NL(T_{A(p_t, \ell_j)})|}{k})^{-1} \right] \end{aligned}$$

□

The proof for Theorem 4.7 is as follows:

Proof. Let P denote the value of the objective function of the primal solution and D denote the value of the objective function of the dual solution. Initially, let $P = 0$ and $D = 0$. In the following, we prove three claims:

- (1) The primal solution produced by the algorithm is feasible.
- (2) The dual solution produced by the algorithm is feasible.
- (3) $P \leq 2 \ln(1 + k)D$.

By three claims and weak duality of linear programs, the theorem follows immediately.

The proof of claim (1) and (2) are similar to that of claim (1) and (2) in section 3.7.

Third, we prove claim (3) as follows. If the algorithm increases the variables $a_{S,t}$ at some time t , then: $\frac{\partial D}{\partial a_{S,t}} = |S| - k$. Let's compute the primal cost. At depth $j \in \{\ell_1 + 1, \ell_2 + 1, \dots, \ell_h + 1\}$, we compute the movement cost of our algorithm by the change of $\nabla b'_j$ as follows.

$$\begin{aligned}
& \sum_{w \in NFC(A(p_t, j-1)) \setminus \{A(p_t, j)\}} 2D(w) \cdot \frac{du_{w,t}}{db'_j} \cdot \frac{db'_j}{a_S} \\
&= \nabla b'_j \sum_{w \in C(A(p_t, j-1)) \setminus \{A(p_t, j)\}} 2D(w) \cdot \frac{\ln(1+k)}{2D(w)} (u_{w,t} + \frac{|NL(T_w)|}{k}) \\
&= \frac{u_{r,t} + \frac{|S|}{k}}{\phi(j)} \cdot \ln(1+k) \cdot \frac{\sum_{w \in C(A(p_t, j-1)) \setminus \{A(p_t, j)\}} (u_{w,t} + \frac{|NL(T_w)|}{k})}{u_{A(p_t, j-1),t} + \frac{|NL(T_{A(p_t, j-1)})|}{k}} \\
&= \frac{\sum_{p \in S} u_{p,t} + \frac{|S|}{k}}{\phi(j)} \cdot \ln(1+k) \cdot \frac{\sum_{w \in C(A(p_t, j-1)) \setminus \{A(p_t, j)\}} (u_{w,t} + \frac{|NL(T_w)|}{k})}{u_{A(p_t, j-1),t} + \frac{|NL(T_{A(p_t, j-1)})|}{k}} \\
&< \frac{(\sum_{p \in S \setminus \{p_t\}} u_{p,t} + \frac{|S|-1}{k} + 2)}{\phi(j)} \cdot \ln(1+k) \cdot \frac{\sum_{w \in C(A(p_t, j-1)) \setminus \{A(p_t, j)\}} (u_{w,t} + \frac{|NL(T_w)|}{k})}{u_{A(p_t, j-1),t} + \frac{|NL(T_{A(p_t, j-1)})|}{k}} \\
&= \frac{(\sum_{p \in S \setminus \{p_t\}} u_{p,t} + \frac{|S|-1}{k} + 2(|S|-k))}{\phi(j)} \cdot \ln(1+k) \cdot \frac{\sum_{w \in C(A(p_t, j-1)) \setminus \{A(p_t, j)\}} (u_{w,t} + \frac{|NL(T_w)|}{k})}{u_{A(p_t, j-1),t} + \frac{|NL(T_{A(p_t, j-1)})|}{k}} \\
&\leq (3(|S| - k) + \frac{(|S|-1)}{k}) \cdot \ln(1+k) \cdot \frac{\sum_{w \in C(A(p_t, j-1)) \setminus \{A(p_t, j)\}} (u_{w,t} + \frac{|NL(T_w)|}{k})}{u_{A(p_t, j-1),t} + \frac{|NL(T_{A(p_t, j-1)})|}{k}} \\
&\leq 4 \cdot \ln(1+k) \cdot (|S| - k) \cdot \frac{\sum_{w \in C(A(p_t, j-1)) \setminus \{A(p_t, j)\}} (u_{w,t} + \frac{|NL(T_w)|}{k})}{u_{A(p_t, j-1),t} + \frac{|NL(T_{A(p_t, j-1)})|}{k}} \\
&\leq 4 \cdot \ln(1+k) \cdot (|S| - k), \text{ since } \sum_{w \in C(A(p_t, j-1)) \setminus \{A(p_t, j)\}} (u_{w,t} + \frac{|NL(T_w)|}{k}) \leq u_{A(p_t, j-1),t} + \frac{|NL(T_{A(p_t, j-1)})|}{k}.
\end{aligned}$$

Where the first inequality holds since $\sum_{p \in S \setminus \{p_t\}} u_{p,t} < |S| - k$, the reason is that the constraint at time t is not satisfied otherwise the algorithm stop increasing the variable $u_{p,t}$ (Since $\sum_{p \in S} u_{p,t} = |S| - k$, $\sum_{p \in S \setminus \{p_t\}} u_{p,t} < |S| - k \Leftrightarrow u_{p_t,t} = 0$, i.e the algorithm stop increasing the variables). In addition, when $|S| \geq k + 1$, $\frac{|S|-1}{k} = |S| - k$.

Thus, the total cost of all j depth is at most $4\ell \cdot \ln(1+k) \cdot (|S| - k)$. Hence, we get $P \leq 4\ell \cdot \ln(1+k) \cdot D$.

So, the competitive ratio of this algorithm is $4\ell \ln(1 + k)$.

□